ON THE FLIPS FOR A SYNCHRONIZED SYSTEM

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Abstract. It is shown that if an infinite synchronized system has a flip, then it has infinitely many non-conjugate flips, and that the result cannot be extended to the class of coded systems.

1. Introduction

Let X be a (two-sided) shift space, so that (X, σ_X) is an invertible topological dynamical system. A homeomorphism $\varphi : X \to X$ is called a *flip* for (X, σ_X) if $\varphi^2 = \mathrm{id}_X$ and $\varphi\sigma_X = \sigma_X^{-1}\varphi$. In this case, we say that (X, σ_X, φ) is a *shift-flip system*. The simplest one may be the reversal map ρ defined by $\rho(x)_i = x_{-i}$, provided X is closed under ρ. Two shift-flip systems (X, σ_X, φ) and (Y, σ_Y, ψ) are said to be *conjugate* if there is a homeomorphism $\Phi : X \to Y$ such that $\Phi \circ \sigma_X = \sigma_Y \circ \Phi$ and $\Phi \circ \varphi = \psi \circ \Phi$. The homeomorphism Φ is called a *conjugacy* from (X, σ_X, φ) to (Y, σ_Y, ψ). Two flips φ and φ' for (X, σ_X) are said to be conjugate if the shift-flip systems (X, σ_X, φ) and (X, σ_X, φ') are conjugate. Thus the flips φ and φ' are conjugate if and only if there is an automorphism θ of (X, σ_X) such that $\theta\varphi = \varphi'\theta$.

Suppose φ is a flip for (X, σ_X). Then the shift dynamical systems (X, σ_X) and (X, σ_X^{-1}) are conjugate, but this does not hold in the general case ([6], Examples 7.4.19 and 12.3.2). Thus not every shift dynamical system has a flip. On the other hand, it is easy to see that the maps $\sigma_X^m\varphi$, $m \in \mathbb{Z}$, are flips for (X, σ_X), and that they are all distinct whenever $|X| = \infty$. Therefore if an infinite shift dynamical system has a flip, then it has infinitely many different ones. But $\sigma_X^m\varphi$ and $\sigma_X^n\varphi$ are conjugate whenever $n - m$ is even, because

$$\sigma_X^{(n-m)/2}\,(\sigma_X^m\varphi) = (\sigma_X^n\varphi)\,\sigma_X^{(n-m)/2}.$$

Although the flips $\sigma_X^m\varphi$, $m \in \mathbb{Z}$, are all distinct, each of them is conjugate to one of the two flips φ and $\sigma_X\varphi$.

In this paper, we will be interested in the property that *if (X, σ_X) has a flip, then it has infinitely many non-conjugate ones.* In [5], it is shown that if $|\mathcal{A}| \geq 2$, then the full $\mathcal{A}$-shift $(\mathcal{A}^{\mathbb{Z}}, \sigma)$ has infinitely many non-conjugate flips. Let M denote the Morse shift [3, 7], that is, M is the set of bi-infinite sequences in $\{0,1\}^{\mathbb{Z}}$ which does not contain any block in the set

$$\{awawa : a \in \{0,1\} \text{ and } w \in \mathcal{B}(\{0,1\}^{\mathbb{Z}})\}.$$

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It is evident that the reversal map ρ and the map $\psi : M \to M$ defined by $\psi(x)_i = 1 - x_{-i}$ are flips for (M, σ_M). Suppose φ is a flip for (M, σ_M). Then $\rho\varphi$ is an automorphism of (M, σ_M). But it is known [1] that the automorphism group of (M, σ_M) is generated by $\{\sigma_M, \rho\psi\}$, and we have $(\rho\psi)^2 = \mathrm{id}_M$; hence φ is conjugate to one of the four flips ρ, ψ, $\sigma_M\rho$ and $\sigma_M\psi$. Thus every infinite full shift has the property, while the Morse shift does not.

The purpose of this paper is to establish the following:

Theorem A. *If X is a synchronized system, $|X| = \infty$ and there is a flip for (X, σ_X), then there are infinitely many non-conjugate flips for (X, σ_X).*

Theorem B. *There is an infinite coded system W such that the reversal map ρ is a flip for (W, σ_W) and $\{\sigma_W^m \rho : m \in \mathbb{Z}\}$ is the set of flips for (W, σ_W).*

In Theorem B, the result implies that every flip for (W, σ_W) is conjugate to one of the two flips ρ and $\sigma_W\rho$, because $\sigma_W^m\rho$ and $\sigma_W^n\rho$ are conjugate whenever $m - n$ is even. We recall that every irreducible sofic shift is a synchronized system and every synchronized system is coded [4, 6].

If φ is a flip for (X, σ_X), we denote the set $\{x \in X : \sigma_X^n(x) = \varphi(x) = x\}$ by $F(\varphi; n)$ for $n = 1, 2, 3, \ldots$, and denote by $A(\varphi)$ the set of points $x \in X$ such that a finitary (intrinsically synchronizing) block appears infinitely often in x and

$$0 < |\{i \in \mathbb{Z} : \varphi(x)_i \neq x_i\}| < \infty.$$

It is clear that if (X, σ_X, φ) and (Y, σ_Y, ψ) are shift-flip systems and Φ is a conjugacy from (X, σ_X, φ) to (Y, σ_Y, ψ), then $\Phi(F(\varphi; n)) = F(\psi; n)$ for all n, and $\Phi(A(\varphi)) = A(\psi)$. It is also clear that $|F(\varphi; n)| < \infty$ for all n. Now, Theorem A is an immediate consequence of the following technical results which are proved in the next section, and an inductive argument.

Proposition C. *If X is an infinite synchronized system and φ is a flip for (X, σ_X), then at least one of the two sets $A(\varphi)$ and $A(\sigma_X\varphi)$ is non-empty.*

Proposition D. *If X is an infinite synchronized system, φ is a flip for (X, σ_X), and $A(\varphi) \neq \emptyset$, then there is a flip ψ for (X, σ_X) such that*

(a) $|F(\varphi; n)| \leq |F(\psi; n)|$ *for all n,*
(b) $|F(\varphi; n)| < |F(\psi; n)|$ *for some n, and*
(c) $A(\psi) \neq \emptyset$.

We prove Theorem B by constructing W explicitly (Section 3). In [2], it is shown that there is a coded system whose automorphism group is generated by the shift map and is isomorphic to $\mathbb{Z}$ ([2], Corollary 2.2), but it is not clear whether the coded system has a flip. Fortunately, we can simplify the construction to obtain a coded system W such that the automorphism group of (W, σ_W) is $\{\sigma_W^m : m \in \mathbb{Z}\}$ and the the reversal map ρ is a flip for (W, σ_W). It is then trivial to see that W has the property stated in Theorem B.

2. Proof of Propositions C and D

We start with some preliminaries. Let X be a shift space and φ be a flip for (X, σ_X). If $\varphi(x)_0 = \varphi(x')_0$ whenever $x_0 = x'_0$, then there is a unique map $\tau : \mathcal{B}_1(X) \to \mathcal{B}_1(X)$ such that

$$\varphi(x)_i = \tau(x_{-i}) \quad (x \in X, i \in \mathbb{Z}),$$

and consequently $\tau^2 = \mathrm{id}_{\mathcal{B}_1(X)}$. In this case, we say that φ is a *one-block* flip and τ is the *symbol map* of φ. The following lemma states that every flip for a shift dynamical system can be recoded to a one-block flip.

Lemma 2.1. *Suppose X is a shift space and φ is a flip for (X, σ_X). Then there are a finite set $\mathcal{A}$, a shift space Y over $\mathcal{A}$, and a one-block flip ψ for (Y, σ_Y) such that (Y, σ_Y, ψ) is conjugate to (X, σ_X, φ).*

Proof. Let $\mathcal{A} = \{(x_0, \varphi(x)_0) : x \in X\}$. For $x = \langle x_i \rangle_{i\in\mathbb{Z}} \in X$ let $\Phi(x)$ denote the bi-infinite sequence $\langle (x_i, \varphi(x)_{-i}) \rangle_{i\in\mathbb{Z}}$ in $\mathcal{A}^{\mathbb{Z}}$, and set $Y = \{\Phi(x) : x \in X\}$. It is then clear that Y is a shift space and Φ is a conjugacy between the shift spaces X and Y. If we put $\psi = \Phi \circ \varphi \circ \Phi^{-1}$, then ψ is a one-block flip for (Y, σ_Y), Φ is a conjugacy from (X, σ_X, φ) to (Y, σ_Y, ψ), and the symbol map is given by $\mathcal{A} \ni (a, b) \mapsto (b, a) \in \mathcal{A}$. □

Suppose φ is a one-block flip and τ is the symbol map. For notational simplicity, we write $\tau(a) = a^*$ for $a \in \mathcal{B}_1(X)$, and if $w = w_1 w_2 \cdots w_n \in \mathcal{B}_n(X)$, we denote the block $w_n^* \cdots w_2^* w_1^*$ by w^*. Thus we have $\varphi(x)_{[i,j]} = \left(x_{[-j,-i]}\right)^*$ for $x \in X$ and for $i \le j$. It is clear that $w^* \in \mathcal{B}(x)$ whenever $w \in \mathcal{B}(X)$, and that $(w^*)^* = w$. It is also clear that if w is finitary, then so is w^*. If N is a positive integer, the map $\varphi^{[N]} : X^{[N]} \to X^{[N]}$ defined by

$$\varphi^{[N]}(y)_i = (y_{-i})^*$$

is a one-block flip for the N-th higher block system $(X^{[N]}, \sigma_{X^{[N]}})$ of (X, σ_X). It is easy to see that if N is odd, then $(X^{[N]}, \sigma_{X^{[N]}}, \varphi^{[N]})$ is conjugate to (X, σ_X, φ), otherwise it is conjugate to $(X, \sigma_X, \sigma_X\varphi)$.

In our proof of the propositions, the following lemma will play a crucial role.

Lemma 2.2. *Suppose that X is an irreducible shift space, $|X| = \infty$, and $f \in \mathcal{B}_1(X)$. Then there are blocks $a, b \in \mathcal{B}(X)$ such that*

(a) *$faf, fbf \in \mathcal{B}(X)$,*
(b) *f does not appear in b, and*
(c) *$fbfa$ does not appear in any $(|a|+1)$-periodic point.*

Proof. Since X is irreducible and $|X| = \infty$, there are blocks a and b such that $faf, fbf \in \mathcal{B}(X)$ and the following hold:

(i) if $fa'f \in \mathcal{B}(X)$, then $|a| \le |a'|$,
(ii) $b \ne (af)^n a$ for all $n \ge 0$, and
(iii) if $fb'f \in \mathcal{B}(X)$ and $b' \ne (af)^n a$ for all $n \ge 0$, then $|b| \le |b'|$.

It is then easy to see that the blocks a and b have the desired properties. □

We recall that a *synchronized system* is an irreducible shift space which has a finitary block.

Lemma 2.3. *Suppose that X is an infinite synchronized system and φ is a flip for (X, σ_X). If there is a point $x \in X$ such that $\varphi(x) = x$ and a finitary block appears in x, then $A(\varphi) \ne \emptyset$.*

Proof. Suppose $x \in X$, $\varphi(x) = x$ and a finitary block appears in x. By Lemma 2.1, we may assume that φ is a one-block flip. Then $(x_{[-n,n]})^* = x_{[-n,n]}$ for all $n \ge 0$. Since a finitary block appears in x, it follows that $x_{[-n,n]}$ is also finitary whenever n becomes sufficiently large. Suppose $x_{[-n,n]}$ is finitary. By passing to the $(2n+1)$-st

higher block system, we may assume that there is a finitary symbol (finitary block of length one) f such that $f^* = f$. Let $a, b \in \mathcal{B}(X)$ satisfy the conditions in Lemma 2.2, and let N be a positive integer such that

$$2|a| + 1 + 2(|b| + 1) \leq (N - 1)(|a| + 1). \tag{2.1}$$

If we write $afa(fb)^2(fa)^{2N} = w$ and $|a|+|b|+1+N(|a|+1) = M$, then $|w| = 2M+1$ and $(a^*f)^k w(fa)^k \in \mathcal{B}(X)$ for all $k \geq 0$. Hence there is a point $y \in X$ such that $y_{(-\infty,-M-1]} = \cdots a^*fa^*fa^*f$, $y_{[-M,M]} = w$ and $y_{[M+1,\infty)} = fafafa\cdots$. It is evident that the finitary symbol f appears infinitely often in y, and $\varphi(y)_i = y_i$ whenever $|i| \geq M + 1$. Since the block $fbfa$ does not appear in any $(|a| + 1)$-periodic point, (2.1) implies that $w^* \neq w$; hence $\varphi(y)_i \neq y_i$ for some $i \in [-M, M]$. This proves the lemma. □

Proof of Proposition C. Suppose X is an infinite synchronized system, φ is a flip for (X, σ_X) and f is a finitary block. By Lemma 2.1, we may assume that φ is a one-block flip. Then f^* is also a finitary block, and there are blocks v and w such that $fvf, f^*wf \in \mathcal{B}(X)$.

We first consider the case when $|w|$ is odd. If we write $|w| = 2N + 1$, then there is a point $x \in X$ such that $x_{(-\infty,-N-1]} = \cdots v^*f^*v^*f^*v^*f^*$, $x_{[-N,N]} = w$ and $x_{[N+1,\infty)} = fvfvfv\cdots$. It is obvious that the finitary block f appears in x infinitely often, $\varphi(x)_i = x_i$ whenever $|i| \geq N + 1$ and that $\varphi(x)_{[-N,N]} = w^*$. If $w^* = w$, then $\varphi(x) = x$, and Lemma 2.3 implies that $A(\varphi) \neq \emptyset$; otherwise we have $\varphi(x) \neq x$, and hence $x \in A(\varphi)$.

Now, suppose that N is a positive integer and $|w| = 2N$. Then there is a point $x \in X$ such that $x_{(-\infty,-N-1]} = \cdots v^*f^*v^*f^*v^*f^*$, $x_{[-N,N-1]} = w$ and $x_{[N,\infty)} = fvfvfv\cdots$. In this case, we have $\sigma_X\varphi(x)_i = x_i$ for all $i \in (-\infty, -N-1] \cup [N, \infty)$, and $\sigma_X\varphi(x) = x$ if and only if $w^* = w$. Thus we have $A(\sigma_X\varphi) \neq \emptyset$, by the same reasoning as in the first case. □

For $A, B \subset \mathbb{Z}$ and $m \in \mathbb{Z}$, we denote the sets $\{mj : j \in A\}$ and $\{j + k : j \in A \text{ and } k \in B\}$ by mA and $A + B$, respectively. We also write $(-1)A = -A$ and $\{m\} + A = m + A$.

Proof of Proposition D. Suppose X is an infinite synchronized system, φ is a flip for (X, σ_X) and $A(\varphi) \neq \emptyset$. We prove the proposition by constructing an automorphism θ of (X, σ_X) and homeomorphisms $\theta_1, \theta_2, \theta_3, \ldots$ from X onto itself such that $\theta\varphi$ is a flip for (X, σ_X), and the following hold:

(i) $\theta_n(F(\varphi; n)) \subset F(\theta\varphi; n)$ for all n,
(ii) $\theta_n(F(\varphi; n)) \neq F(\theta\varphi; n)$ for some n, and
(iii) $A(\theta\varphi) \neq \emptyset$.

By Lemma 2.1, we may assume that φ is a one-block flip. Let f be a finitary block. We may assume that $|f|$ is odd. By passing to the $|f|$-th higher block system, we may assume that f is a finitary symbol. Let $a, b \in \mathcal{B}(X)$ satisfy the conditions in Lemma 2.2. Since X is irreducible and $A(\varphi) \neq \emptyset$, there is a block c of odd length such that $f^*cf \in \mathcal{B}(X)$ and $c^* \neq c$. Let N be a positive integer such that

$$|a| + 2|b| + |c| + 2 \leq (N - 1)(|a| + 1), \tag{2.2}$$

and put $fb(fa)^N = d$. Then d and d^* are finitary blocks, and $d^*cd, d^*c^*d \in \mathcal{B}(X)$. For notational simplicity, we write $|c| = 2\alpha + 1$ and $\alpha + |d| = \beta$, so that $|d^*cd| = |d^*c^*d| = 2\beta + 1$. Since the symbol f does not appear in the block b,

and since the block $fbfa$ does not appear in any $(|a|+1)$-periodic point, the inequality (2.2) implies that the following statement holds: If $x \in X$, $i \neq j$, and $x_{[i-\beta,i+\beta]}, x_{[j-\beta,j+\beta]} \in \{d^*cd, d^*c^*d\}$, then $|i-j| \geq |c|+|d|+1$. For $x \in X$ let $\mathcal{M}(x)$ denote the set of integers i such that $x_{[i-\beta,i+\beta]} \in \{d^*cd, d^*c^*d\}$. Then we have

$$[i-\alpha-1, i+\alpha+1] \cap [j-\beta, j+\beta] = \emptyset \qquad (i,j \in \mathcal{M}(x),\ i \neq j) \tag{2.3}$$

for every $x \in X$. In particular, the intervals $[i-\alpha, i+\alpha]$, $i \in \mathcal{M}(x)$, are mutually disjoint. For each $i \in \mathcal{M}(x)+[-\alpha,\alpha]$ let $c(i;x)$ denote the unique element of $\mathcal{M}(x)$ such that $i \in [c(i;x)-\alpha, c(i;x)+\alpha]$.

For $A \subset \mathbb{Z}$ and $x \in X$ we define the bi-infinite sequence $\theta_A(x)$ by

$$\theta_A(x)_i = \begin{cases} (x_{2c(i;x)-i})^* & \text{if } \ i \in (\mathcal{M}(x) \cap A) + [-\alpha,\alpha], \\ x_i & \text{otherwise.} \end{cases}$$

Thus θ_A replaces the part $x_{[i-\alpha,i+\alpha]}$ of x with $(x_{[i-\alpha,i+\alpha]})^*$ whenever $i \in \mathcal{M}(x) \cap A$ and leaves the remaining part of x unchanged. Since d and d^* are finitary blocks and $d^*cd, d^*c^*d \in \mathcal{B}(X)$, we have $\theta_A(x) \in X$ for all $x \in X$. From this and (2.3), it follows that $\mathcal{M}(x) = \mathcal{M}(\theta_A(x))$, and consequently $\theta_A(\theta_A(x)) = x$ for all $x \in X$. If $x, x' \in X$, $i \in \mathbb{Z}$ and $x_{[i-\alpha-\beta,i+\alpha+\beta]} = x'_{[i-\alpha-\beta,i+\alpha+\beta]}$, then $\theta_A(x)_i = \theta_A(x')_i$. Hence $\theta_A : X \to X$ is a homeomorphism satisfying $\theta_A^2 = \mathrm{id}_X$ for every $A \subset \mathbb{Z}$. It is easy to see that $\theta_A\theta_B = \theta_{(A \triangle B)}$, $\sigma_X\theta_A = \theta_{(-1+A)}\sigma_X$ and $\varphi\theta_A = \theta_{(-A)}\varphi$.

We define θ to be $\theta_{\mathbb{Z}}$. It is clear that θ is an automorphism of (X,σ_X) such that $\theta^2 = \mathrm{id}_X$ and $\theta\varphi = \varphi\theta$. In particular, the map $\theta\varphi$ is a flip for (X,σ_X). For $n = 1,2,3,\ldots$ we set

$$H(n) = \bigcup_{k \in \mathbb{Z}} \left\{ i \in \mathbb{Z} : nk < i < n\left(k + \frac{1}{2}\right) \right\},$$

and define θ_n to be $\theta_{H(n)}$.

To prove (i), suppose n is a positive integer and $x \in F(\varphi;n)$, that is, $\sigma_X^n(x) = \varphi(x) = x$. We have $n + H(n) = H(n)$, hence $\sigma_X^n(\theta_n(x)) = \theta_n(x)$. If we put $C = \mathbb{Z} \setminus (H(n) \cup (-H(n)))$, then $\{H(n), -H(n), C\}$ is a partition of $\mathbb{Z}$. We have $C = n\mathbb{Z}$ in the case when n is odd, and $C = n\mathbb{Z} + \{0, n/2\}$ in the case when n is even. Since $\sigma_X^n(x) = \varphi(x) = x$ and $c^* \neq c$, it follows that $\mathcal{M}(x) \cap C = \emptyset$. Hence we have

$$\mathcal{M}(x) \cap (\mathbb{Z} \triangle (-H(n))) = \mathcal{M}(x) \cap (H(n) \cup C) = \mathcal{M}(x) \cap H(n),$$

so that

$$\theta\varphi(\theta_n(x)) = \theta\theta_{(-H(n))}\varphi(x) = \theta_{(\mathbb{Z}\triangle(-H(n)))}(x) = \theta_n(x).$$

Thus $\theta_n(x) \in F(\theta\varphi;n)$, and (i) is proved.

To prove (ii) and (iii), we first construct a point in X. Since X is irreducible, there is a block w such that $dwd^* \in \mathcal{B}(X)$. If we write $2(|c|+2|d|+|w|) = n$ and $|c|+2|d|+n = 2m+1$, then we have $n = |w^*d^*cdwd^*cd|$, and there is a point $z \in X$ such that $\sigma_X^n(z) = z$ and

$$z_{[-m,m]} = d^*cdw^*d^*cdwd^*cd.$$

We have $z_{[-\alpha,\alpha]} = c$, and this implies that $\varphi(z) \neq z$, because $c^* \neq c$. Hence $z \notin F(\varphi;n)$. Since $\sigma_X^n(z) = z$, we have $\sigma_X^n(\theta_n(z)) = \theta_n(z)$. Let C be as in the proof of (i). Then $C = n\mathbb{Z} + \{0, n/2\}$ and we have $\varphi(z) = \theta_C(z)$, hence

$$\theta\varphi(\theta_n(z)) = \theta\theta_{(-H(n))}\varphi(z) = \theta\theta_{(-H(n))}\theta_C(z) = \theta_n(z).$$

Thus $\theta_n(z) \in F(\theta\varphi; n)$, while $z \notin F(\varphi; n)$. Since θ_n is one-to-one, we see that (ii) holds. Finally, we have $\theta\varphi(\theta_n(z)) = \theta_n(z)$, and the finitary block d appears in $\theta_n(z)$, hence (iii) follows from Lemma 2.3. This proves the proposition. □

3. Proof of Theorem B

We recall that a *coded system* X is a shift space which has a code, that is, a set $\mathcal{C}$ of blocks such that the set of bi-infinite concatenations of blocks from $\mathcal{C}$ is dense in X. It is clear that for every set $\mathcal{C}$ of blocks over a finite alphabet there is a unique coded system for which $\mathcal{C}$ is a code. In this section, we follow the method given in Section 1 of [2] to construct an infinite coded system W such that W is closed under the reversal map ρ and its automorphism group is $\{\sigma_W^m : m \in \mathbb{Z}\}$.

Let $\mathcal{A} = \{0, 1, 2\}$, $I = \bigcup_{k \geq 1}[2^{2k}, 2^{2k+1}] = [4, 8] \cup [16, 32] \cup \cdots$, and $J = \{(0, 0), (1, 1), (1, 2), (2, 1), (2, 2)\}$. In [2], a block $w \in \mathcal{B}(\mathcal{A}^{\mathbb{Z}})$ is defined to be *stable* if

(1) the blocks 12 and 21 do not appear in w, and
(2) if $x \in \mathcal{A}^{\mathbb{Z}}$, $x_{[1,3|w|+2]} = 0^{|w|+1}w0^{|w|+1}$, $1 \leq n \leq |w|$, $a \in \{1, 2\}$, $|w|+1 \leq i < j \leq 2|w|+2$, and $x_{[i,j]} = 0a^n0$, then $a = 1$ if and only if $(n, (x_{i-n}, x_{j+n})) \in (I \times J) \cup (I^C \times J^C)$.

We set $\mathcal{C} = \{0\} \cup \{0^{|w|+1}w0^{|w|+1} : w \text{ is stable}\}$, and define W to be the coded system for which $\mathcal{C}$ is a code. The following are easy consequences of the definitions:

(a) For each integer $j \neq 0$ the sets $\{n : n, n+j \in I\}$, $\{n : n \in I \text{ and } n+j \notin I\}$, $\{n : n \notin I \text{ and } n+j \in I\}$ and $\{n : n, n+j \notin I\}$ are all infinite.
(b) 0^n is stable for all n, 1^n is stable if and only if $n \in I$, and 2^n is stable if and only if $n \notin I$.
(c) If $w = w_1w_2 \ldots w_{|w|}$ is stable, then so is the reversed block $w_{|w|} \ldots w_2w_1$.
(d) If w is stable, then $0^\infty.w0^\infty \in W$.
(e) If w, w' are stable and $n \geq \max\{|w|, |w'|\}$, then $w0^{n+1}w'$ is stable. In particular, every finite concatenation of blocks from $\mathcal{C}$ is stable.
(f) $x \in W$ if and only if for every $n \geq 0$ there is a stable block w such that $x_{[-n,n]}$ is a subblock of w.
(g) Suppose $x \in W$. Then the blocks 12 and 21 do not appear in x. If $i < j$, $a \in \{1, 2\}$, $n > 0$ and $x_{[i,j]} = 0a^n0$, then $a = 1$ if and only if $(n, (x_{i-n}, x_{j+n})) \in (I \times J) \cup (I^C \times J^C)$.

Let W_0 denote the set of $x \in W$ such that $|\{i \in \mathbb{Z} : x_i \neq 0\}| < \infty$. For $x \in W$ we set $\mathcal{Z}(x) = \{i \in \mathbb{Z} : x_i = 0\}$. By (d), (f) and (g), we have the following:

(h) W_0 is a dense subset of W. If $x, x' \in W_0$ and $\mathcal{Z}(x) = \mathcal{Z}(x')$, then $x = x'$.

First of all, it is obvious that $|W| = \infty$. By (c) and (f), W is closed under the reversal map ρ. If φ is a flip for (W, σ_W), then $\rho\varphi$ is an automorphism of (W, σ_W). Hence Theorem B is an immediate consequence of the following.

Proposition 3.1. *The automorphism group of* (W, σ_W) *is* $\{\sigma_W^m : m \in \mathbb{Z}\}$.

Remark 3.2. This proposition is a simplified version of Proposition 1.6 in [2], and is proved by essentially the same reasoning as in that paper. Nevertheless, we present the proof here for the readers convenience.

Proof. By (h), it is enough to show that for every automorphism θ of (W, σ_W) there is an integer m such that

$$\mathcal{Z}(\sigma_W^m \theta(x)) = \mathcal{Z}(x) \qquad (x \in W_0). \tag{3.1}$$

Suppose θ is an automorphism of (W, σ_W). Then there is a non-negative integer N and there is a block map $\Theta : \mathcal{B}_{2N+1}(W) \to \mathcal{B}_1(W)$ such that

$$\theta(x)_i = \Theta(x_{[i-N,i+N]}) \qquad (x \in W,\ i \in \mathbb{Z}). \tag{3.2}$$

It is clear that $0^\infty, 1^\infty, 2^\infty \in W$ and $\theta(\{0^\infty, 1^\infty, 2^\infty\}) = \{0^\infty, 1^\infty, 2^\infty\}$.

We first show that $\theta(0^\infty) = 0^\infty$. Suppose, to obtain a contradiction, that $\theta(0^\infty) \neq 0^\infty$. Then there are $a, b \in \{1, 2\}$ such that $\theta(a^\infty) = 0^\infty$ and $\theta(0^\infty) = b^\infty$. Let M be a positive integer such that a^M is stable and $M \geq 2N + 1$. By (d) and (e), we have

$$0^\infty.a^M 0^n a^M 0^\infty \in W$$

whenever $n \geq M + 1$. Since the blocks 12 and 21 do not appear in any $x \in W$, $\theta(a^\infty) = 0^\infty$, $\theta(0^\infty) = b^\infty$, and since $M \geq 2N + 1$, (3.2) implies that there are blocks u, v and integers j, k such that $j \geq -2N$ and

$$\sigma_W^k \theta(0^\infty.a^M 0^n a^M 0^\infty) = b^\infty u.0b^{n+j}0vb^\infty$$

for all $n \geq M+1$. If we denote the right hand side by y, then we have $y_{[0,n+j+1]} = 0b^{n+j}0$. Suppose $n + j > \max\{|u|, |v|\}$. Then $(y_{-(n+j)}, y_{2(n+j)+1}) = (b, b) \in J$, hence (g) implies that $b = 1$ in the case when $n+j \in I$, and $b = 2$ in the case when $n + j \notin I$. But $n + j \in I$ for infinitely many n, and also $n + j \notin I$ for infinitely many n. This contradiction shows that $\theta(0^\infty) = 0^\infty$.

Since $\theta(0^\infty) = 0^\infty$, we have $\theta(W_0) = W_0$ and $\theta(1^\infty) = c^\infty$ for some $c \in \{1, 2\}$. By (b) and (d), $0^\infty.01^n0^\infty \in W$ for all $n \in I$, and there are blocks p, q and integers l, m such that $l \geq -2N$ and

$$\sigma_W^m \theta(0^\infty.01^n 0^\infty) = 0^\infty p.0c^{n+l}0q0^\infty$$

for all $n \in I$ with $n \geq 2N + 1$. There are infinitely many $n \in I$ such that $n + l \in I$, but if $l \neq 0$ then there are also infinitely many $n \in I$ such that $n + l \notin I$. By the same reasoning as in the proof of $\theta(0^\infty) = 0^\infty$, we conclude that $l = 0$ and $c = 1$. Thus we have

$$\sigma_W^m \theta(0^\infty.01^n 0^\infty) = 0^\infty p.01^n 0q0^\infty \qquad (n \in I,\ n \geq 2N + 1). \tag{3.3}$$

To prove (3.1), suppose that $x \in W_0$ and $x_0 = 0$. Let L be a positive integer such that $L \geq N + |m|$ and $x_i = 0$ for $|i| > L$. Let $n \in I$ be such that $n \geq \max\{3L + 2, |q| + 1\}$. Then $x_{[-L,-1]}0x_{[1,L]}0^{n-L}1^n$ is stable,

$$z = 0^\infty x_{[-L,-1]}.0x_{[1,L]}0^{n-L}1^n0^\infty \in W_0,$$

$\sigma_W^m \theta(x)_0 = \sigma_W^m \theta(z)_0$, and (3.3) implies that

$$\sigma_W^m \theta(z)_{[n,3n+1]} = 01^n 0q0^{n-|q|}.$$

In particular, $\sigma_W^m \theta(z)_{[n,2n+1]} = 01^n0$ and $\sigma_W^m \theta(z)_{3n+1} = 0$. Since $n \in I$, (g) implies that $\sigma_W^m \theta(z)_0 = 0$. Hence $\sigma_W^m \theta(x)_0 = 0$. Since $\sigma_W(W_0) = W_0$, we conclude that

$$\mathcal{Z}(x) \subset \mathcal{Z}(\sigma_W^m \theta(x)) \qquad (x \in W_0).$$

If we apply this result to θ^{-1}, we see that there is an integer m' such that

$$\mathcal{Z}(x) \subset \mathcal{Z}(\sigma_W^{m'} \theta^{-1}(x)) \qquad (x \in W_0).$$

We then have

$$\mathcal{Z}(x) \subset \mathcal{Z}(\sigma_W^m \theta(x)) \subset \mathcal{Z}(\sigma_W^{m'} \theta^{-1} \sigma_W^m \theta(x)) = \mathcal{Z}(\sigma_W^{m'+m}(x)) \qquad (x \in W_0),$$

but this implies that $m' + m = 0$, and we obtain (3.1). $\square$

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